# Pyramid Flexagons

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## Introduction

Traditionally, the leaves used to construct flexagons are flat convex polygons that are hinged together in a band [1]. The leaves are often regarded as rigid, although leaf bending is sometimes allowed during flexing. However, there is no mathematical reason why leaves have to be flat. The only restriction on their shape is that adjacent leaves must nest without interference when folded together. Collapsoids, described in 1975 by Pedersen [2, 3], are non convex polyhedra, with several edges left disconnected, that they can be collapsed flat. Collapsoids are made by replacing individual rhombic faces of polyhedra by open based square pyramids, with edges hinged together. The pyramids point inwards. The motions of the faces of a collapsoid, as it is collapsed, are vaguely reminiscent of the motions of some flexagons. Similarly, a pyramid flexagon is a conventional flexagon with the flat leaves replaced by appropriate open based right pyramids. Three examples are described below.

## Pyramid Trihexaflexagon

The net for the pyramid trihexaflexagon is shown in Figure 1. This was derived by replacing the equilateral triangles of a trihexaflexagon [1] by triangular open base pyramids made from 45°-45°-90° triangles. The flexagon works well if made from 160 gsm card with the short edges of the triangles about 5 cm long. To assemble the flexagon, make three copies of Figure 1, and cut along the heavy lines. Then crease the lines between triangles to form hinges, transfer the numbers and asterisk in brackets to the lower face, and delete from the upper face. Next, assemble each part of the net. Overlap and glue together triangles to form open pyramids that point alternately up and down. There are two different ways of doing this. All three parts of the net must be assembled in the same way so that they are identical. Complete the net by gluing together two pairs of triangles marked with asterisks so that faces with asterisks are adjacent. At this stage the net should look like Figure 2(a). Next, fold together adjacent pyramids numbered 3; one of each pair of pyramids goes inside the other. Finally, glue together the two triangles marked with an asterisks so that faces with asterisks are adjacent. As assembled the flexagon should look like Figure 2(b).

As assembled the flexagon is in main position 2(1) with pyramids numbered 2 on the upper face and pyramids numbered 1 on the lower face. As for the hexaflexagon, it can be flexed around a 3-cycle in which pairs of pyramid numbers appear in cyclic order. To do this pinch together pairs of pats (pairs of pyramids or single pyramids) to reach an intermediate position (Figure 2(c)). This can be done in four ways, only two of which work. Then open the flexagon into a new main position, and so on round the 3-cycle.

## Pyramid Fundamental Square Flexagon

The net for the pyramid fundamental square flexagon is shown in Figure 3. This was derived by replacing the squares of a first order fundamental square flexagon [1] by square pyramids made from 54°-54° -72° triangles. The flexagon works well if made from origami duo paper, as in Figure 4, with the short edges of the triangles about 4 cm long. To assemble the flexagon make two copies of Figure 3, and cut along the heavy lines. Then crease the lines between triangles to form hinges, transfer the numbers and asterisk in brackets to the lower face, and delete from the upper face. Next, assemble each part of the net. Overlap and glue together triangles to form open pyramids that point alternately up and down. There are two different ways of doing this. Both parts of the net must be assembled in the same way so that they are identical. The flexagon is easier to assemble if the asterisks are inside the pyramids. Complete the net by gluing together a pair of triangles marked with asterisks so that faces with asterisks are adjacent. At this stage the net should look like Figure 4(a). Next, fold together adjacent pyramids numbered 3 and 4; one of each pair of pyramids goes inside the other. Finally, glue together the two triangles marked with an asterisks so that faces with asterisks are adjacent. As assembled the flexagon should look like Figure 4(b).

As assembled the flexagon is in main position 1(2) with pyramids numbered 1 on the upper face and pyramids numbered 2 on the lower face. As for the first order fundamental square flexagon, it can be flexed around a 4-cycle in which pairs of pyramid numbers appear in cyclic order. To do this fold the flexagon in two to reach an intermediate position (Figure 4(c)). This can be done in four ways only two of which work. Then open the flexagon into a new main position, and so on round the 4-cycle. An intermediate position can also be opened into the equivalent of a box position [1], as shown in Figure 4(d). The pyramids of a main position can be collapsed to reach a collapsed main position (Figure 4(e)), so it can be described as a collapsoid flexagon. From a given main position this can be done in two different ways. There are four different main positions, so there are 8 different collapsed main positions. The flexagon can be flexed into numerous other positions. It is easy to get it into a tangle, from which it is difficult to return it to a main position.

**Pyramid Fundamental Pentagon Flexagon**

The net for a pyramid fundamental pentagon flexagon is shown in Figure 5. This was derived by replacing the regular pentagons of one of the first order fundamental pentagon flexagons [1] by pentagonal pyramids made from equilateral triangles. The flexagon works well if made from 160m gsm card with the edges of the equilateral triangles about 4½ cm long. To assemble the flexagon make two copies of Figure 5, and cut along the heavy lines. Then crease the lines between triangles to form hinges, transfer the numbers and asterisk in brackets to the lower face, and delete from the upper face. Next, assemble each part of the net. Overlap and glue together triangles to form pyramids that point alternately up and down. There are two different ways of doing this. Both parts of the net must be assembled in the same way so that they are identical. The flexagon is easier to assemble if the asterisks are inside the pyramids. Complete the net by gluing together a pair of triangles marked with asterisks so that faces with asterisks are adjacent. At this stage the net should look like Figure 6(a). Next, fold together adjacent pyramids numbered 3, 4 and 5; one of each pair of pyramids goes inside the other. Finally, glue together the two triangles marked with an asterisks so that faces with asterisks are adjacent. As assembled the flexagon should look like Figure 6(b).

As assembled the flexagon is in principal main position 2(1) with pyramids numbered 2 on the upper face and pyramids numbered 1 on the lower face. As for the first order fundamental pentagon flexagon, it can be flexed around a principal 5-cycle in which pairs of pyramid numbers appear in cyclic order. To do this fold the flexagon in two to reach an intermediate position (Figure 6(c)). This can be done in four ways only two of which work. Then open the flexagon into a new main position, and so on round the principal 5-cycle. The continuous path when flexing between adjacent intermediate positions is aesthetically satisfying. An intermediate position can also be opened into a subsidiary main position, as shown in Figure 6(d). The corresponding subsidiary 5-cycle cannot be traversed directly, it has to be traversed via principal main positions.

**Discussion**

In making these pyramid flexagons it was found that the height of the pyramids, choice of material, size of the flexagon, and method of assembly were all fairly critical for satisfactory flexing. The pyramid flexagons chosen illustrate three possibilities. The pyramids of the pyramid trihexaflexagon are rigid. The pyramids of the pyramid fundamental square flexagon can be collapsed flat, so the flexagon is a collapsoid flexagon. This is only possible when the pyramids have an even number of faces. The pyramids of the fundamental pentagon flexagon are not rigid, but have an odd number of faces so cannot be collapsed flat.

## References


1 POOK L P. Serious fun with flexagons. A compendium and guide. Dordrecht: Springer, 2009.

2 PEDERSEN J J. Collapsoids. Mathematical Gazette, 1975, **59**(408), 81-94.

3 HILTON P and PEDERSEN J. A mathematical tapestry. Cambridge: Cambridge University Press, 2010.

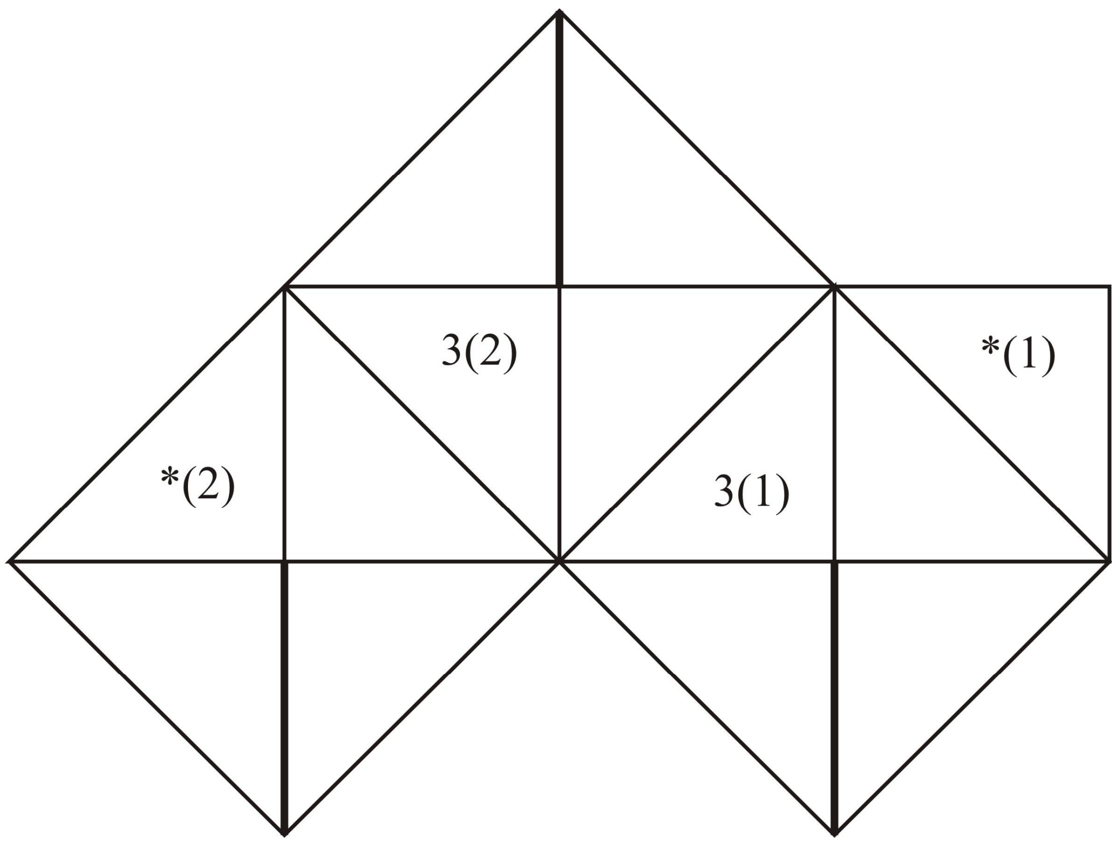




Figure 1. Net for the pyramid trihexaflexagon.

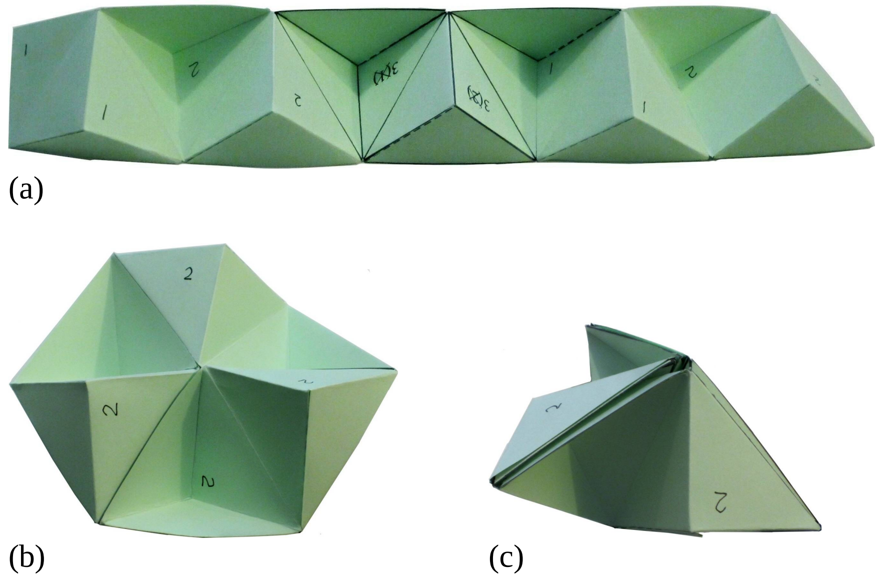



Figure 2. Pyramid trihexaflexagon. (a) Assembled net. (b) Main position. (c) Intermediate position.

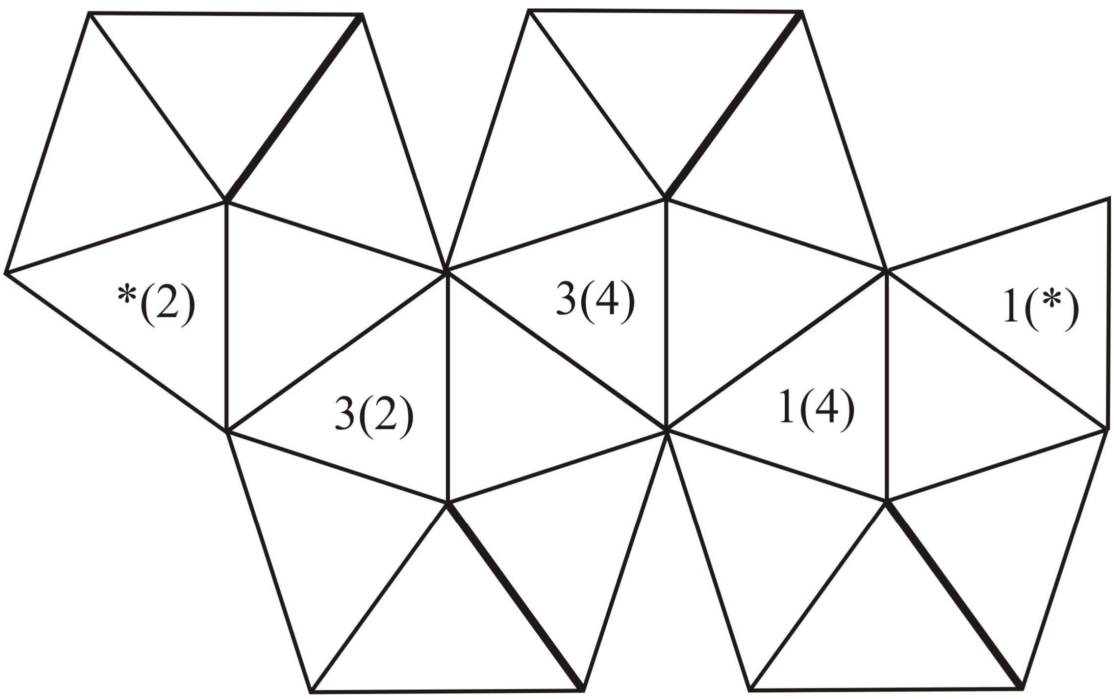



Figure 3. Net for the pyramid fundamental square flexagon.

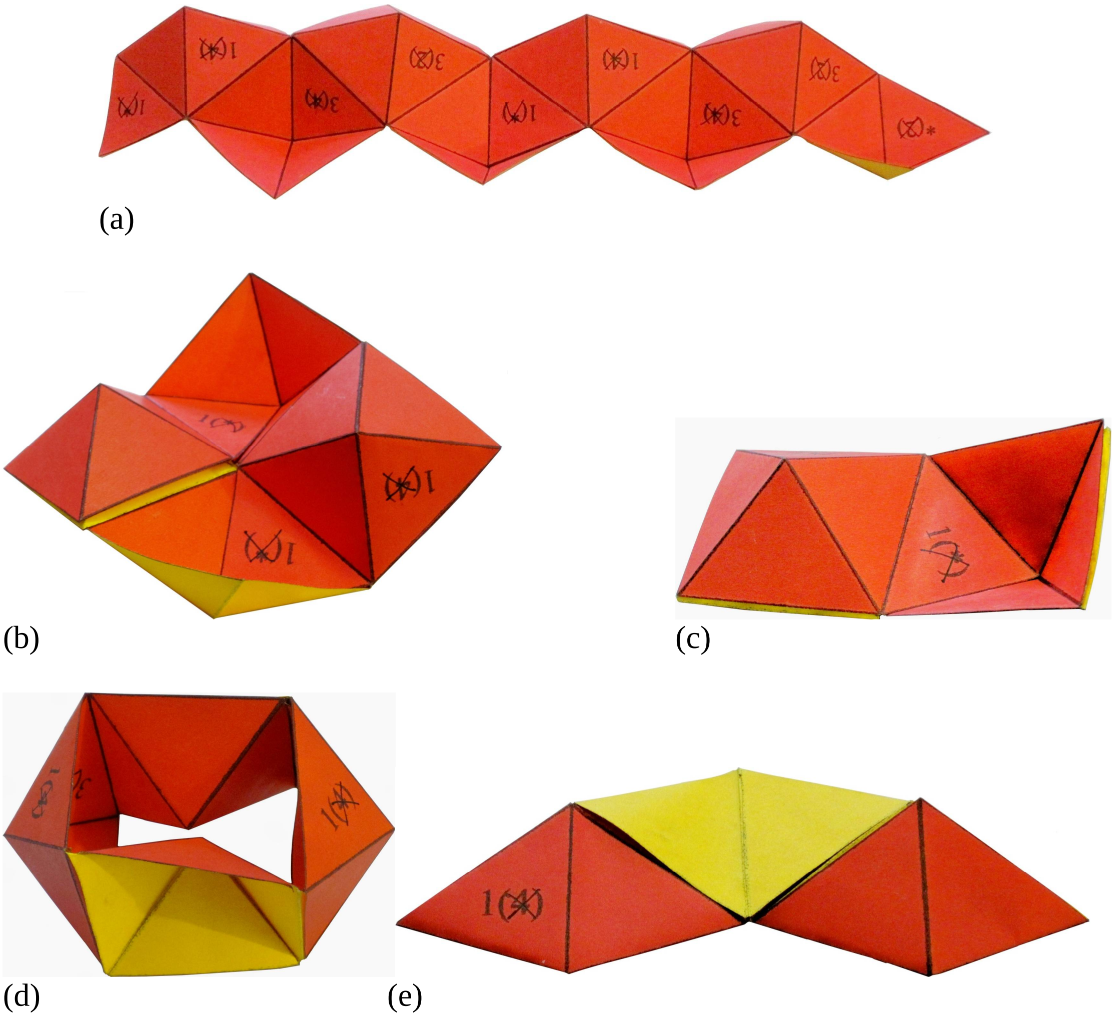



Figure 4. The pyramid fundamental square flexagon. (a) Assembled net. (b) Main position. (c) Intermediate position. (e) Collapsed main position.

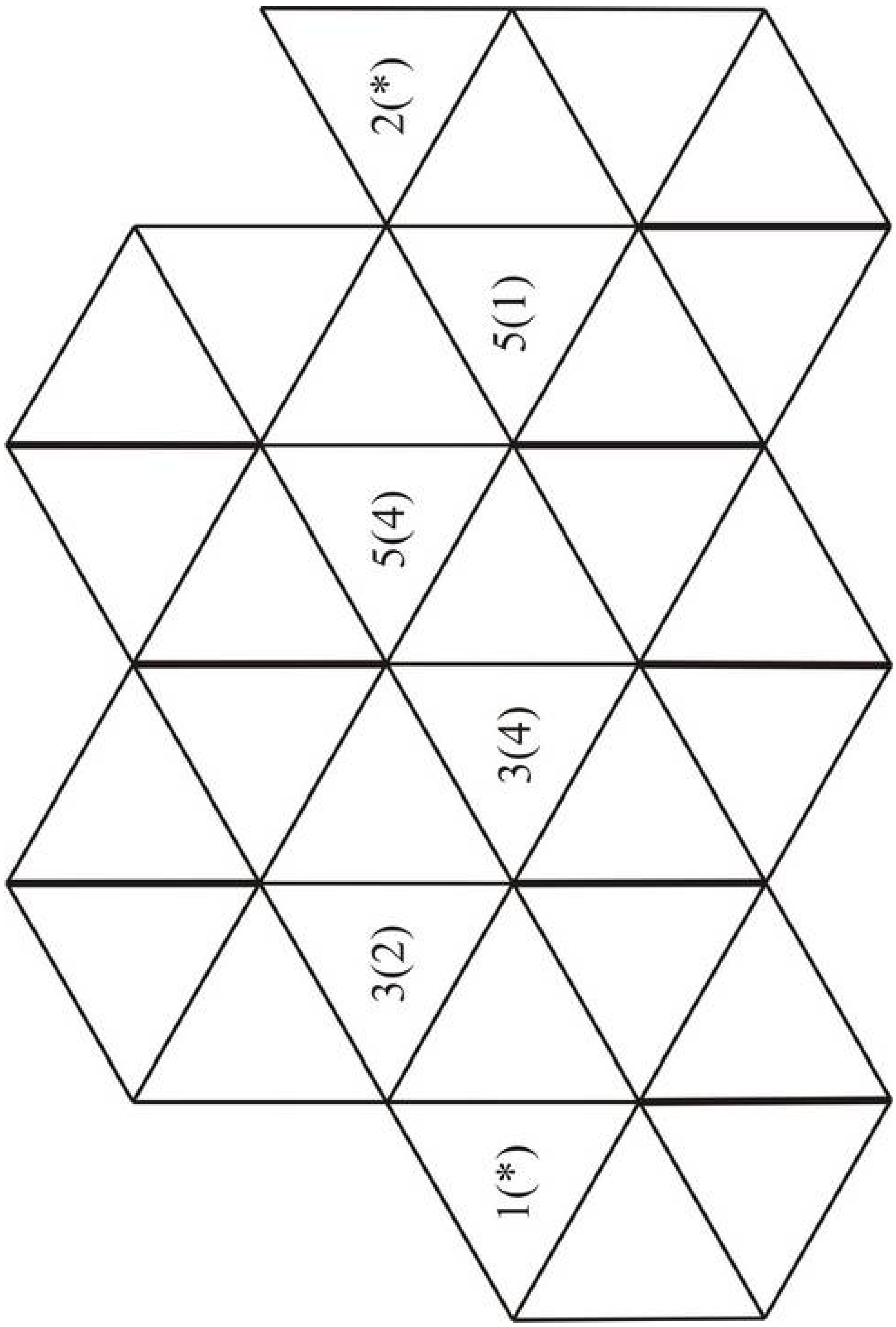
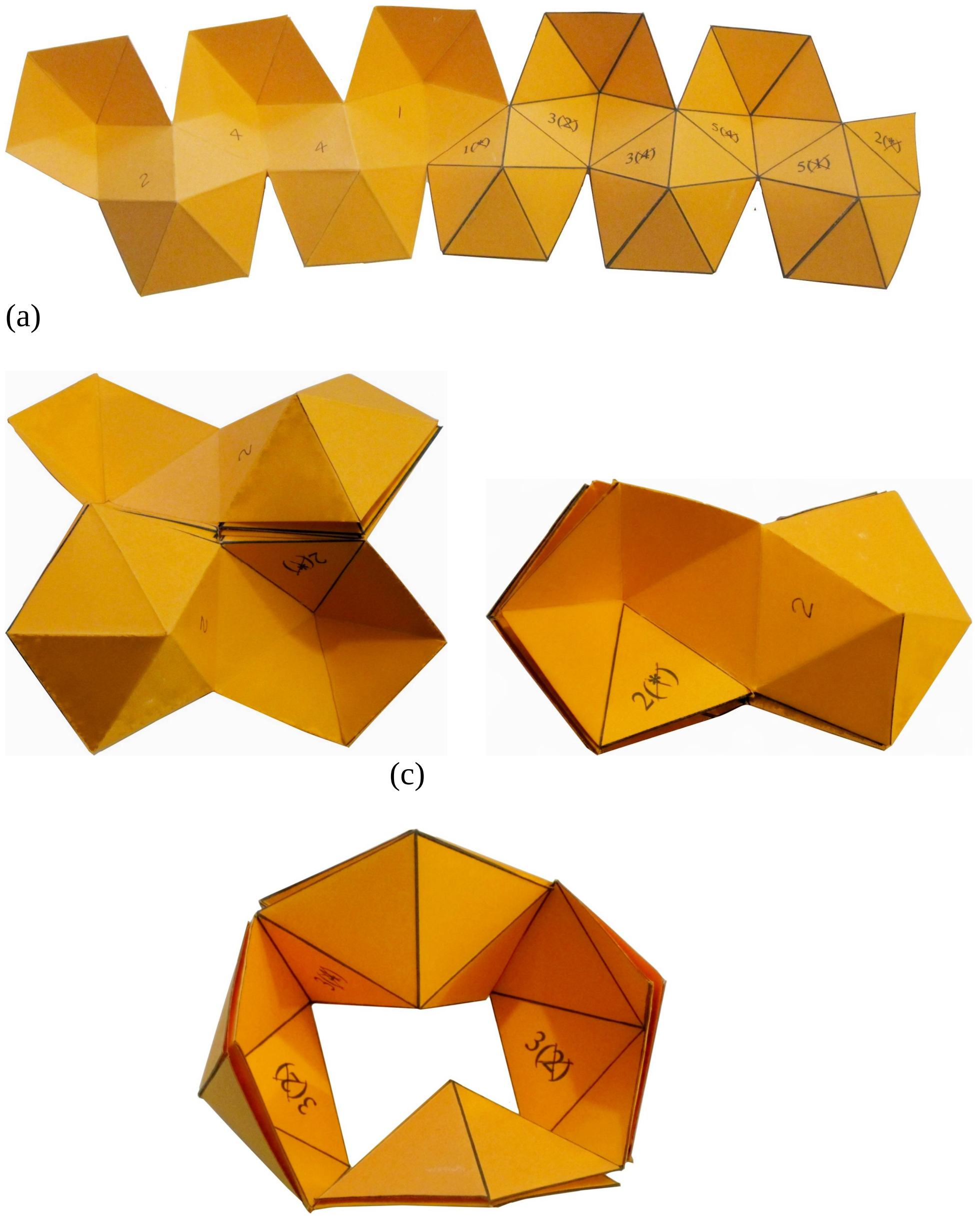



Figure 5. Net for a pyramid fundamental pentagon flexagon.



(a) (b) (c) (d)

Figure 6. Pyramid fundamental pentagon flexagon. (a) Assembled net. (b) Principal main position. (c) Intermediate position. (d) Subsidiary main position.